Deriving the geometric albedo for any reflection law

$$A_g = 2\int_0^1 \rho_0 \, \mu^2 \, d\mu$$

evaluated at zero phase

Examples:

$P_0 = 1$ (Isotropic)

$P_0 = \frac{3}{2}$ (Rayleigh)

$$\Longrightarrow A_g = \frac{\omega}{8}\left(P_0 - 1\right) + \frac{\epsilon}{2} + \frac{\epsilon^2}{6} + \frac{\epsilon^3}{24}$$

single scattering — multiple scattering (isotropic)

$$\epsilon = \frac{1-\gamma}{1+\gamma} \qquad \gamma = \sqrt{1-\omega}$$

Heng, Morris & Kitzmann (2021, Nature Astronomy, 5, 1001)

Deriving the integral phase function for any reflection law

$$\Psi = \frac{12\rho_S\Psi_S + 16\rho_L\Psi_L + 9\rho_C\Psi_C}{12\rho_{S0} + 16\rho_L + 6\rho_C}$$

Reflection law is contained here

Physically, it has three terms:

1. Single-scattering-like behavior
2. Lambertian-sphere-like behavior
3. Correction term

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Deriving the integral phase function for any reflection law

$$\Psi = \frac{12\rho_S\Psi_S + 16\rho_L\Psi_L + 9\rho_C\Psi_C}{12\rho_{S0} + 16\rho_L + 6\rho_C}$$

$$\begin{aligned}
\rho_S &= P_\star - 1 + \frac{1}{4}(1+\epsilon)^2(2-\epsilon)^2,\\
\rho_{S0} &= P_0 - 1 + \frac{1}{4}(1+\epsilon)^2(2-\epsilon)^2,\\
\rho_L &= \frac{\epsilon}{2}(1+\epsilon)^2(2-\epsilon),\\
\rho_C &= \epsilon^2(1+\epsilon)^2.
\end{aligned}$$

$$\begin{aligned}
\Psi_S &= 1 - \frac{1}{2}\left[\cos\left(\frac{\alpha}{2}\right) - \sec\left(\frac{\alpha}{2}\right)\right]\Psi_0,\\
\Psi_L &= \frac{1}{\pi}\left[\sin\alpha + (\pi-\alpha)\cos\alpha\right],\\
\Psi_C &= -1 + \frac{5}{3}\cos^2\left(\frac{\alpha}{2}\right) - \frac{1}{2}\tan\left(\frac{\alpha}{2}\right)\sin^3\left(\frac{\alpha}{2}\right)\Psi_0,\\
\Psi_0 &= \ln\left[\frac{(1+\alpha_-)(\alpha_+-1)}{(1+\alpha_+)(1-\alpha_-)}\right],\\
\alpha_\pm &= \sin\left(\frac{\alpha}{2}\right) \pm \cos\left(\frac{\alpha}{2}\right).
\end{aligned}$$

Tedious, yes, but completely closed-form (analytical)!

Heng, Morris & Kitzmann (2021, Nature Astronomy, 5, 1001)

The most important step of the derivation…

The reflection law (scattering phase function) may be taken out of the integral as well!

$$F_S = \frac{\omega\rho_S}{4}\int_0^{\pi/2}\cos^2\Theta \, d\Theta \int_{\alpha-\pi/2}^{\pi/2}\frac{\cos\Phi \, \cos(\alpha-\Phi)}{\cos\Phi + \cos(\alpha-\Phi)}\, d\Phi$$

$$F_L = \frac{\omega\rho_L}{4}\int_0^{\pi/2}\cos^3\Theta \, d\Theta \int_{\alpha-\pi/2}^{\pi/2}\cos\Phi \, \cos(\alpha-\Phi)\, d\Phi$$

$$F_C = \frac{\omega\rho_C}{4}\int_0^{\pi/2}\cos^4\Theta \, d\Theta \int_{\alpha-\pi/2}^{\pi/2}\frac{\cos^2\Phi \, \cos^2(\alpha-\Phi)}{\cos\Phi + \cos(\alpha-\Phi)}\, d\Phi$$

$$\Psi = \frac{F}{F_0}$$

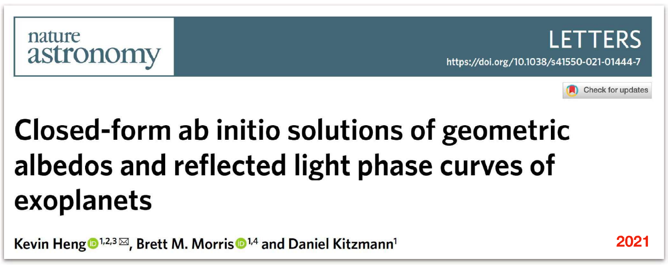
evaluated at zero phase

Notice how the integrals may be done separately in the observer-centric coordinate system!

Heng, Morris & Kitzmann (2021, Nature Astronomy, 5, 1001)

Executive summary and implications of *Heng, Morris & Kitzmann (2021)*

- Derived analytical, ab initio solutions for geometric albedo and integral phase function for *any* scattering law.
- Allows the user to specify any scattering phase function.
- One approximation made: multiple scattering occurs isotropically. Holds for scattering asymmetry factors of $g \lesssim 0.5$.
- Built on insights from Russell (1916), Chandrasekhar (1960), Sobolev (1975) and Hapke (1981).
- Ab initio solutions have no tuning parameters, implying that they can be used to extract fundamental parameters from data.


nature astronomy — LETTERS
https://doi.org/10.1038/s41550-021-01444-7
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Closed-form ab initio solutions of geometric albedos and reflected light phase curves of exoplanets

Kevin Heng[1,2,3], Brett M. Morris[1,4] and Daniel Kitzmann[1]

2021

Approximate, but general, solution to a mathematical problem in classical astronomy, first posed by Russell (1916)